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ALGORITHMIC APPROACH TO OPTIMAL RESOURCE CONTROL IN AN OPEN ECONOMY MODEL

Abstract. The dynamic development of the global economy and the impact of external factors create challenges for ensuring sustainable growth, making effective resource management an urgent task. This paper addresses the problem of optimal resource management in an open economy model with constraints on labor distribution, investment, and foreign trade balances. To solve this problem, a numerical algorithm for optimal control in an open economic system has been developed, with simultaneous consideration of labor, investment, and foreign trade constraints, ensuring the construction of admissible controls and the recovery of economically interpretable indicators while preserving balance relations. The proposed approach transforms the original constrained problem into a form convenient for sequential numerical implementation and feedback control synthesis. The conducted computational experiments confirmed the effectiveness of the approach: the constructed state and control trajectories demonstrate the achievement of a stable equilibrium state of the system while satisfying all imposed constraints. The practical value of the study lies in the fact that the algorithm provides the possibility of implementing numerical solutions to optimal control problems and using them for analyzing and forecasting economic development under resource limitations.

Keywords: numerical methods, optimal control algorithms, open systems modeling, resource allocation, software tools, simulation and analysis.

1. Introduction

Taking into account the dynamic development of the global economy and the influence of external factors, achieving sustainable growth is of particular importance for open economic systems. However, external factors such as fluctuations in the global market, trade restrictions and dependence on imports of investment goods significantly affect the economic stability of the country, creating additional risks to economic stability. These issues highlight the need to develop economic and mathematical models that can optimize resource allocation, taking into account external constraints and enabling the forecasting of economic dynamics to support long-term stability and growth.

Another important challenge in this area is the difficulty of obtaining analytical solutions for optimal control problems in nonlinear economic systems. This stimulates the development of numerical methods, computational algorithms, and specialized software environments for solving such problems [1], [2]. A significant line of research therefore focuses on creating efficient computational procedures and decision-support tools for dynamic controlled systems.

One of the key tools for analyzing economic dynamics is the Cobb-Douglas production function, which allows us to model the interaction of factors of production and assess their impact on economic growth. A wide range of economic-mathematical models describing the dynamics of economic systems based on the Cobb-Douglas production function has been presented in the literature. For example, the study published on [3] examines a balanced growth model of an open three-sector economy and analyzes the conditions for sustainable development [4] develop a three-factor model of enterprise dynamics with internal and external investments, described by a system of nonlinear differential equations. In [5], a two-sector economic growth model with time delays is investigated, and bifurcation effects are analyzed.

In the modern literature, a variety of numerical approaches have been developed for solving constrained optimal control problems, including projection techniques and sequential convex programming methods [6]–[8]. These approaches are primarily aimed at computational efficiency and at the treatment of general dynamic systems. In contrast, the present study focuses on linking the control synthesis procedure with the structure of

macroeconomic balance relations of an open economy.

In contrast to previously studied closed-economy models [9]–[13], the present work extends the optimal control framework to an open economic system with explicit consideration of foreign trade flows and industrial security constraints. The proposed approach links the control synthesis procedure with the structure of macroeconomic balance relations, enabling the computed control variables to be translated into investment, labor, and foreign trade proportions.

2. Materials and Methods

2.1. Mathematical Model of an Open Economic System

Let us consider the mathematical model of an open economy and denote the subsystems of the model as follows: $i = 0$ – production subsystem, $i = 1$ – investment subsystem, $i = 2$ – consumer subsystem. According to this model, the output volume in each i -th subsystem is determined by the Cobb–Douglas production function:

$$X_i = F_i(K_i, L_i) = A_i K_i^{\alpha_i} L_i^{1-\alpha_i}, \quad (1)$$

$(i = 0, 1, 2)$

where X_i – output volume, K_i – volume of fixed productive assets, L_i – volume of labor resources, A_i – coefficient of neutral technological progress, α_i – capital elasticity coefficient, $(1 - \alpha_i)$ –labor elasticity coefficient.

Let us introduce the following notations:

$\theta_i = \frac{L_i}{L}$ – shares in labor resource distribution across subsystems;

$s_i = \frac{I_i}{X_1}$ – shares in investment resource distribution across subsystems;

$f_i(k_i) = \frac{X_i}{L_i}$ – labor productivity in the i -th subsystem.

$k_i = \frac{K_i}{L_i}$ – capital-labor ratio of the subsystems;

$\delta_1 = \frac{Y_1}{L}$ – share of imported goods for investment;

$\delta_2 = \frac{Y_2}{L}$ – share of imported goods for consumption;

$x_i = \theta_i f_i(k_i)$ – specific output of the subsystems.

Then, the Equation (1) can be rewritten in the following form for the capital-labor ratio of the subsystems:

$$\begin{aligned} \dot{k}_i(t) &= -\rho_i k_i(t) + \\ &+ \left(\frac{s_i(t)}{\theta_i(t)} \right) (x_1 + \delta_1), \end{aligned} \quad (2)$$

$$k_i(0) = k_i^0, \quad \rho_i > 0,$$

$$\begin{aligned} x_i &= \theta_i(t) A_i k_i(t)^{\alpha_i}, \\ A_i &> 0, \quad 0 < \alpha_i < 1. \end{aligned} \quad (3)$$

$i = 0, 1, 2.$

with balance equations:

Investment balance:

$$\begin{aligned} s_0(t) + s_1(t) + s_2(t) &= 1, \\ s_i(t) &> 0, \quad i = 0, 1, 2 \end{aligned} \quad (4)$$

Labor balance:

$$\begin{aligned} \theta_0(t) + \theta_1(t) + \theta_2(t) &= 1, \\ \theta_i(t) &> 0, \quad i = 0, 1, 2 \end{aligned} \quad (5)$$

Material balance:

$$\begin{aligned} (1 - \beta_0)x_0 &= \beta_1 x_1 + \beta_2 x_2 + \delta_0, \\ 0 < \beta_i < 1, \quad i &= 0, 1, 2 \end{aligned} \quad (6)$$

Foreign trade balance:

$$\mu_0 \delta_0 = \mu_1 \delta_1 + \mu_2 \delta_2 \quad (7)$$

Industrial security:

$$\delta_1 \leq \gamma_1 x_1, \quad \delta_2 \leq \gamma_2 x_2 \quad (8)$$

Where γ_1 – import quota coefficient for investment goods, δ_0 – share of material exports, μ_0 – world price of exported materials, μ_1, μ_2 – world prices of imported investment and consumer goods, ρ_i – coefficient of capital-labor ratio reduction due to capital depreciation and employment growth, β_i – direct material costs per unit of output in the i -th subsystem.

A numerical algorithm for finding the stable state of an open economic system, based on the Lagrange multipliers method and the golden section principle, was proposed in [14].

2.2. Formulation of the Problem for Optimal Resource Allocation

In this study, industrial security is interpreted as ensuring conditions that maintain equilibrium and stability within the system:

$$\delta_1 = \gamma_1 x_1, \delta_2 = \gamma_2 x_2. \quad (9)$$

Then, the equation for the foreign trade balance Equation (7) can be rewritten as follows:

$$\delta_0 = \frac{\mu_1}{\mu_0} \gamma_1 x_1 + \frac{\mu_2}{\mu_0} \gamma_2 x_2. \quad (10)$$

The initial problem (2)–(8), consisting of six controls, can be transformed into a problem with three controls using the balance equations. To achieve this, we rewrite the deviations from the equilibrium state of the system of differential equations (2), introducing the following notations:

$$\begin{aligned} y_0(t) &= k_0(t) - k_0^s, y_1(t) = \\ &= k_1(t) - k_1^s, y_2(t) = k_2(t) - k_2^s. \end{aligned} \quad (11)$$

$$u_0(t) = \frac{s_0(t)\theta_1(t)}{\theta_0(t)} - \frac{s_0^s\theta_1^s}{\theta_0^s}, \quad (12)$$

$$\begin{aligned} u_1(t) &= s_1(t) - s_1^s, u_2(t) = \\ &= \frac{s_2(t)\theta_1(t)}{\theta_2(t)} - \frac{s_2^s\theta_1^s}{\theta_2^s}. \end{aligned}$$

As a result of the transformations performed, system (2) takes the following form:

$$\begin{aligned} \dot{y}_0(t) &= -\rho_0(y_0(t) + k_0^s) + \\ &+ \left(u_0(t) + \frac{s_0(t)\theta_1(t)}{\theta_0(t)} \right) \times \\ &(1 + \gamma_1)A_1(y_1(t) + k_1^s)^{\alpha_1}, \\ y_0(t_0) &= y_0^0, y_0(T) = 0, \end{aligned} \quad (13)$$

$$\begin{aligned} \dot{y}_1(t) &= -\rho_1(y_1(t) + k_1^s) + \\ &+ (u_1(t) + s_1^s) \times \\ &(1 + \gamma_1)A_1(y_1(t) + k_1^s)^{\alpha_1}, \\ y_1(t_0) &= y_1^0, y_1(T) = 0, \end{aligned} \quad (14)$$

$$\begin{aligned} \dot{y}_2(t) &= -\rho_2(y_2(t) + k_2^s) + \\ &+ \left(u_2(t) + \frac{s_2(t)\theta_1(t)}{\theta_2(t)} \right) \times \\ &(1 + \gamma_1)A_1(y_1(t) + k_1^s)^{\alpha_1}, \end{aligned} \quad (15)$$

$$y_2(t_0) = y_2^0, y_2(T) = 0.$$

In the system, let $y(t) = (y_0(t), y_1(t), y_2(t))^*$ – denote the state vector, and $u(t) = (u_0(t), u_1(t), u_2(t))^*$ – denote the control vector.

To obtain a numerical solution of the nonlinear system (13)–(15), we employ the iterative quasilinearization method proposed by Bellman and Kalaba. At each iteration, the right-hand side of the differential equations (13)–(15) is approximated by linear forms derived from a Taylor expansion in a neighborhood of the solution obtained at the previous iteration.

As a result, after a sufficient number of iterations, a sequence of linear–quadratic (LQ) problems is generated, whose solutions approximate the optimal control of the original nonlinear problem. The iterative process is continued until the stopping condition $\|u^{(n)} - u^{(n-1)}\| \leq \varepsilon$ is satisfied, where ε is a prescribed tolerance.

At the n -th iteration, the control $u^{(n)}(t)$ and the corresponding state trajectory $y^{(n)}(t)$ are determined as the solution of the LQ problem constructed in a neighborhood of the previous approximation $(y^{(n-1)}(t), u^{(n-1)}(t))$ [15].

Linearize the system of equations (13)–(15) and obtain the following system of differential equations:

$$\begin{aligned} \dot{y}_0(t) &= \\ &= (1 + \gamma_1)A_1 \frac{s_0(t)\theta_1(t)}{\theta_0(t)} \alpha_1 k_1^{s\alpha_1-1} y_1(t) - \\ &-\rho_0 y_0(t) + (1 + \gamma_1)A_1 k_1^{s\alpha_1} u_0(t) + \\ &+ f(y_1, u_0), y_0(t_0) = y_0^0. \end{aligned} \quad (16)$$

$$\begin{aligned} \dot{y}_1(t) &= \\ &= (-\rho_1 + (1 + \gamma_1)A_1 \alpha_1 k_1^{s\alpha_1-1} s_1^s) y_1(t) + \\ &+ (1 + \gamma_1)A_1 k_1^{s\alpha_1} u_1(t) + \\ &+ f(y_1, u_1), y_1(t_0) = y_1^0, \end{aligned} \quad (17)$$

$$\begin{aligned} \dot{y}_2(t) &= \\ &= (1 + \gamma_1)A_1 \frac{s_2(t)\theta_1(t)}{\theta_2(t)} \alpha_1 k_1^{s\alpha_1-1} y_1(t) - \\ &-\rho_2 y_2(t) + (1 + \gamma_1)A_1 k_1^{s\alpha_1} u_2(t) + \\ &+ f(y_1, u_2), y_2(t_0) = y_2^0, \end{aligned} \quad (18)$$

where the function $f(y_1, u_i)$, $i = 0, 1, 2$, is defined as follows:

$$\begin{aligned}
 & f(y_1, u_0) = \\
 & = (1 + \gamma_1)A_1 \left[u_0(t) \left((y_1(t) + k_1^s)^{\alpha_1} - k_1^{s\alpha_1} \right) + \frac{s_0^s \theta_1^s}{\theta_0^s} \left((y_1(t) + k_1^s)^{\alpha_1} - k_1^{s\alpha_1} - \alpha_1 k_1^{s\alpha_1-1} y_1(t) \right) \right], \\
 f(y_1, u_1) & = (1 + \gamma_1)A_1 \left[u_1(t) \left((y_1(t) + k_1^s)^{\alpha_1} - k_1^{s\alpha_1} \right) + s_1^s \left((y_1(t) + k_1^s)^{\alpha_1} - k_1^{s\alpha_1} - \alpha_1 k_1^{s\alpha_1-1} y_1(t) \right) \right], \\
 & f(y_1, u_2) = \\
 & (1 + \gamma_1)A_1 \left[u_2(t) \left((y_1(t) + k_1^s)^{\alpha_1} - k_1^{s\alpha_1} \right) + \frac{s_2^s \theta_1^s}{\theta_2^s} \left((y_1(t) + k_1^s)^{\alpha_1} - k_1^{s\alpha_1} - \alpha_1 k_1^{s\alpha_1-1} y_1(t) \right) \right],
 \end{aligned}$$

When $y_1(t) = 0$ and $u_i(t) = 0$, the function $f(y_1, u_i), i = 0, 1, 2$, is equal to zero.

Next, we rewrite the given system (16)–(18) in vector–matrix form and consider the optimal control problem for a class of nonlinear controlled systems defined as follows:

$$\begin{aligned}
 \dot{y}(t) & = A(t)y(t) + B(t)u(t) + \\
 & + f(y, u), \quad y(t_0) = y_0, y(T) = 0, \quad (19)
 \end{aligned}$$

$$\begin{aligned}
 u \in U & = \{u \mid \sigma_1 \leq u(t) \leq \sigma_2\}, \quad (20) \\
 & \sigma_1 < 0, \sigma_2 > 0, \sigma_1, \\
 & \sigma_2 = \text{const}, t \in [t_0, T].
 \end{aligned}$$

Let $y(t)$ denote the $(n \times 1)$ state vector of the controlled object and $u(t)$ the $(m \times 1)$ control vector. Let $f(y, u)$ be an $(n \times 1)$ vector of bounded continuous functions on the time interval $t \in [t_0, T]$. σ_1, σ_2 are vectors of dimension $(m \times 1)$. The matrices $A(t), B(t)$ are given time-dependent matrices of dimensions $(n \times n)$ and $(n \times m)$, respectively. While t_0 and T are the prescribed initial and terminal times, respectively. It is assumed that system (19) is controllable at time t_0 .

We define the control quality by the functional:

$$\begin{aligned}
 & J(y, u) = \\
 & = \frac{1}{2} \int_{t_0}^T [y^*(t)Q(t)y(t) + u^*(t)R(t)u(t)]dt \quad (21)
 \end{aligned}$$

where $Q(t)$ positive definite symmetric matrix and matrix $R(t)$ is diagonal and positive-definite. $Q(t)$ represents the “penalty” for deviations of the economic state from equilibrium, while $R(t)$

reflects the costs associated with resource reallocation.

Problem Statement. It is required to construct a control $u(t)$ that transfers the economic system (19) from the initial state $y(t_0) = y_0$ to the desired final state $y(T) = 0$. At the same time, the control must satisfy the bilateral constraints (20) and ensure the minimization of the performance functional (21).

2.3. Solution to the Problem for Optimal Resource Allocation

To solve the stated problem, a modified Lagrange multipliers method is applied. The main idea of this method is to transform the original constrained problem into an equivalent unconstrained problem, the solution of which automatically satisfies the initial constraints.

For the implementation of the method, the system of differential equations (19) is multiplied by the following multiplier $\lambda_0(y, t)$ and use to the functional the terms enforcing the constraints:

$$\begin{aligned}
 & \lambda_1^*(y, t)(\sigma_1 - u(t)) + \\
 & + \lambda_2^*(y, t)(u(t) - \sigma_2) + \\
 & + \lambda_3^*(y, t)(y(t) - W(t, T)^{-1}q(t)), \\
 & \lambda_i(y, t) \geq 0, (i = 1, 2, 3).
 \end{aligned}$$

As a result, we obtain a modified functional that enables the problem to be studied in an unconstrained form while still enforcing the original constraints:

$$\begin{aligned}
 L(u, y) = & \\
 = \int_{t_0}^T & \left[\frac{1}{2} y^*(t) Q(t) y(t) + \frac{1}{2} u^*(t) R(t) u(t) + \lambda_0^*(y, t) (A(t) y(t) + B(t) u(t) + f(y, u) - \dot{y}(t)) \right. \\
 & + \lambda_1^*(y, t) (\sigma_1 - u(t)) + \lambda_2^*(y, t) (u(t) - \sigma_2) \\
 & \left. + \lambda_3^*(y, t) (y(t) - W^{-1}(t, T) q(t)) \right] dt
 \end{aligned} \tag{22}$$

Let us consider the integrand of the functional (22) in the form of the following functions:

$$v(y, t) = \frac{1}{2} y^*(t) K(t) y(t) + y^*(t) q(t), \quad \frac{\partial v}{\partial y} = K(t) y(t) + q(t), \tag{23}$$

$$\begin{aligned}
 M(u, y, t) = & \frac{1}{2} y^*(t) (Q(t) + \dot{K}(t)) y(t) + \frac{1}{2} u^*(t) R(t) u(t) + \\
 + & (K(t) y(t) + q(t))^* (A(t) y(t) + B(t) u(t) + f(y, u)) + y^*(t) \dot{q}(t) + \lambda_1^*(y, t) (\sigma_1 - u(t)) \\
 + & \lambda_2^*(y, t) (u(t) - \sigma_2) + \lambda_3^*(y, t) (y(t) - W^{-1}(t, T) q(t))
 \end{aligned} \tag{24}$$

Thus, the Lagrangian functional (22) can be written in the expanded form as follows:

$$\begin{aligned}
 L(u, y) = & v(y(t_0), t_0) - v(y(T), T) + \\
 + & \int_{t_0}^T M(u, y, t) dt.
 \end{aligned} \tag{25}$$

To solve the formulated problem, we first apply the first-order necessary condition for an extremum and derive the expression for the optimal control in the following form:

$$\begin{aligned}
 u(t) = & \\
 -R^{-1}(t) & B^*(t) (K(t) y(t) + q(t)) + \\
 + R^{-1}(t) & (\lambda_1^*(y, t) - \lambda_2^*(y, t)).
 \end{aligned} \tag{26}$$

Here, $K(t)$ is an $n \times n$ matrix satisfying a matrix Riccati differential equation. The matrices $K(t)$, $W(t)$, and the vector $q(t)$ on the interval $t \in [t_0, T]$ satisfy the following system of differential equations:

$$\begin{aligned}
 \dot{K}(t) + K(t)A(t) + A^*(t)K(t) - \\
 - K(t)B_1(t)K(t) + Q(t) = \\
 = 0, K(t_0) = K_0,
 \end{aligned} \tag{27}$$

$$\begin{aligned}
 \dot{W}(t, T) = W(t, T)A_1^*(t) + \\
 + A_1(t)W(t, T) - B_1(t), \quad W(T, T) = 0,
 \end{aligned} \tag{28}$$

$$\begin{aligned}
 \dot{q}(t) = -A_1^*(t)q(t) + \\
 + W^{-1}(t, T)B(t)\varphi(y, t) + \\
 + W^{-1}(t, T)f(y, \varphi), \\
 q(t_0) = W^{-1}(t_0, T)y(t_0)
 \end{aligned} \tag{29}$$

where:

$$A_1(t) = A(t) - B(t)R^{-1}(t)B^*(t)K(t),$$

$$B_1(t) = B(t)R^{-1}(t)B^*(t),$$

$$\varphi(y, t) = R^{-1}(t)(\lambda_1^*(y, t) - \lambda_2^*(y, t)).$$

The inverse of $W(t, T)$ is required only for $t < T$, whereas at the terminal time the boundary conditions are imposed directly and no inversion is performed.

The solvability of the Riccati-type equations is considered under the standard assumptions of controllability and stabilizability of the pair $(A(t), B(t))$. The weighting matrices $Q(t)$ and $R(t)$ are assumed to be symmetric and positive definite. Under these assumptions, the Riccati equation admits a bounded solution, and the resulting feedback control ensures convergence of the system to the desired terminal state. If these conditions are violated, numerical instability or loss of convergence may occur.

For the multipliers $\lambda_1(y, t) \geq 0, \lambda_2(y, t) \geq 0$, which are nonnegative by definition, the complementarity condition holds, expressed as follows:

$$\begin{aligned}
 \lambda_1^*(y, t) (\sigma_1 - u(t)) = \\
 = 0, \lambda_2^*(y, t) (u(t) - \sigma_2) = 0.
 \end{aligned}$$

For this purpose, they are chosen as follows:

$$\begin{aligned}
 & \lambda_1(y, t) = \\
 = & \max \{0; \sigma_1 - \omega(y, t)\} \geq 0, \lambda_2(y, t) \quad (30) \\
 & = \max \{0; \omega(y, t) - \sigma_2\} \geq 0, \\
 & \omega(y, t) = \\
 = & -R^{-1}(t)B^*(t)(K(t)y(t) + q(t)).
 \end{aligned}$$

Because the matrix $R(t)$ is diagonal, the minimization problem with respect to the control becomes separable by components.

Hence, the complementarity-based construction of the multipliers leads to the componentwise projection of the nominal control $\omega(y, t)$ onto the admissible interval $[\alpha, \beta]$. As a consequence, the control constraints are satisfied at every time instant.

Suppose that solutions to equations (27) and (28) exist and that the conditions (30) are satisfied. Then, under the control law of the form (26), the dynamics of system (19) are described as follows:

$$\begin{aligned}
 \dot{y}(t) = & A_1(t)y(t) - B_1(t)q(t) + \\
 & + B(t)\varphi(y, t) + f(y, \varphi, t), y(t_0) = y_0. \quad (31)
 \end{aligned}$$

Thus, the use of solutions of differential equations (31) and (29) shows that the dynamics of

system (31), corresponding to control law (26), at the final moment of time reaches the state $y(T) = 0$.

To verify compliance with the balance constraints, the following expressions are computed. Since $s_1(t) = u_1(t) + s_1^s$, the remaining portion of investment resources allocated to the production and consumer subsystems is equal to $1 - s_1(t)$. Let $v(t)$ denote the share of the consumer subsystem in this remainder of investment resources. Then, in order to ensure the fulfillment of the investment balance condition (4), the following allocation rule is applied:

$$\begin{aligned}
 s_1(t) &= u_1(t) + s_1^s, \\
 s_0(t) &= v(t)(1 - s_1(t)), \\
 s_2(t) &= (1 - v(t))(1 - s_1(t)). \quad (32)
 \end{aligned}$$

To ensure the fulfillment of the labor balance condition (5), using the notations introduced in (12) we obtain the expressions for $\theta_0(t)$ and $\theta_2(t)$, while the expression for $\theta_1(t)$ is derived from (5) in the following form:

$$\begin{aligned}
 \theta_1(t) &= \frac{1}{1 + \frac{s_0(t)}{u_0(t) + (s_0^s \theta_1^s / \theta_0^s)} + \frac{s_2(t)}{u_2(t) + (s_2^s \theta_1^s / \theta_2^s)}}, \\
 \theta_0(t) &= \frac{v(t)(1 - s_1(t))\theta_1(t)}{u_0(t) + (s_0^s \theta_1^s / \theta_0^s)}, \\
 \theta_2(t) &= \frac{(1 - v(t))(1 - s_1(t))\theta_1(t)}{u_2(t) + \left(\frac{s_2^s \theta_1^s}{\theta_2^s}\right)}. \quad (33)
 \end{aligned}$$

Next, taking into account equation (10) and the obtained values of $\theta_0(t)$ and $\theta_2(t)$ from equation

(33), the material balance equation (6) yields the following expression for $v(t)$:

$$\begin{aligned}
 v(t) = & \frac{\left(\beta_1 + \frac{\mu_1}{\mu_0} \gamma_1\right) A_1 k_1(t)^{\alpha_1} + \left(\beta_2 + \frac{\mu_2}{\mu_0} \gamma_2\right) A_2 k_2(t)^{\alpha_2} \frac{1 - u_1(t) - s_1^s}{u_2(t) + \left(\frac{s_1^s \theta_1^s}{\theta_2^s}\right)}}{\left(1 - \beta_0\right) A_0 (k_0(t))^{\alpha_0} \frac{1 - u_1(t) - s_1^s}{u_0(t) + \left(\frac{s_0^s \theta_1^s}{\theta_0^s}\right)} + \left(\beta_2 + \frac{\mu_2}{\mu_0} \gamma_2\right) A_2 (k_2(t))^{\alpha_2} \frac{1 - u_1(t) - s_1^s}{u_2(t) + \left(\frac{s_2^s \theta_1^s}{\theta_2^s}\right)}}. \quad (34)
 \end{aligned}$$

Algorithm 1: Algorithm for Solving the Problem of Optimal Resource Allocation

- 1: Using the Runge–Kutta method on the interval $t \in [t_0, T]$ integrate the systems of differential equations (27) and (28) subject to the boundary conditions.
 - 2: For the given initial state $y(t_0) = y_0$ we determine the vector $q(t_0)$ according to the relation: $q(t_0) = W^{-1}(t_0, T)y(t_0)$. We integrate the system of equations (31) and (29) over the interval $[t_0, T]$ using the Runge–Kutta method under the
 - 3: initial conditions $y(t_0) = y_0, q(t_0) = q_0$. During the integration process, the trajectory $y(t)$ and the control $u(t)$ are constructed.
Using the values of the effective trajectory $y(t)$ and the effective control $u(t)$, obtained during the integration process,
 - 4: we compute the following expressions:
To verify the fulfillment of the material balance (6), expression (34) is evaluated.
To verify the fulfillment of the investment balance (4), expression (32) is evaluated.
To verify the fulfillment of the labor balance (5), expression (33) is evaluated.
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If, in the course of computations, the initial and terminal conditions of the problem are modified and new numerical experiments are required, the calculations should be repeated starting from the second and third steps of the algorithm.

3. Results

To obtain a numerical solution to the formulated optimal control problem, a series of computational experiments was carried out. When constructing the plots of the effective trajectory $y(t)$ and the optimal control $u(t)$, the model parameter values were taken according to the data presented in Tables 1 and 2.

Table 1. Model coefficients

i	0	1	2
α_i	0,7	0,62	0,45
β_i	0,39	0,49	0,52
λ_i	0,05	0,05	0,05
A_i	3,19	6	3,71

The values of the coefficients $\alpha_i, \beta_i, \rho_i$ and A_i are chosen within the permissible intervals corresponding to the logic of the model. The parameters are not based on specific empirical measurements but serve for the numerical analysis of the system behavior under various configurations [14].

In the numerical experiment, the planning horizon was set to $T = 12$. The initial state of the

system and the control constraints were specified as follows:

$$y(t_0) = (4000, -5500, 4500)^*, \quad (35)$$

Table 2. Stationary state of an open economic system

i	0	1	2
θ_i	0.233	0.270	0.296
s_i	0.391	0.316	0.292
k_i	59433.45	41525.48	20840.3
x_i	1638.1	1181.31	161.75
y_i	335.77	590.66	80.88

To avoid possible singularities in the expressions for $\theta_i(t)$ and $v(t)$, the admissible control set is implemented numerically using a projection procedure.

$$0.1 - u_i^s \leq u_i(t) \leq 0.9 - u_i^s. \quad (36)$$

$$-0.3531 \leq u_0 \leq 0.4469, -0.216 \leq u_1 \leq 0.584, -0.108 \leq u_2 \leq 0.741. \quad (37)$$

The computed values of the deviations of the system state at the final time are as follows (Figure 1):

$$\begin{aligned} y_0(T) &= 5.0475 \cdot 10^{-9}, \\ y_1(T) &= -1.0885 \cdot 10^{-8}, \\ y_2(T) &= -7.1596 \cdot 10^{-9}. \end{aligned}$$

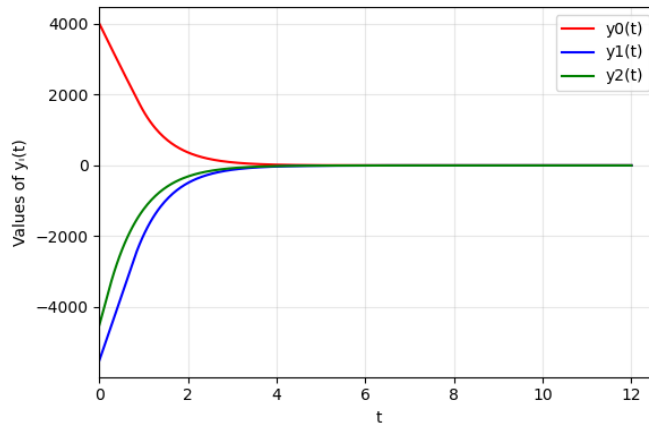


Figure 1. Effective trajectories $y_i(t)$ of the system under optimal control

Figure 2 presents the graph of the optimal control trajectory remains within the admissible region U , defined by the constraints (37). The graph shows that the constructed

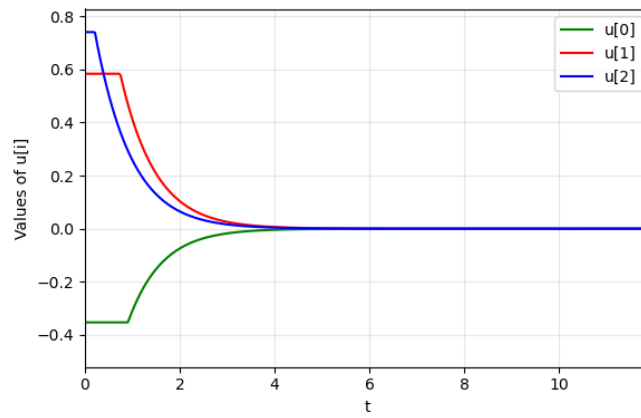


Figure 2. Dynamics of optimal control functions $u_i(t)$ satisfying bilateral constraints

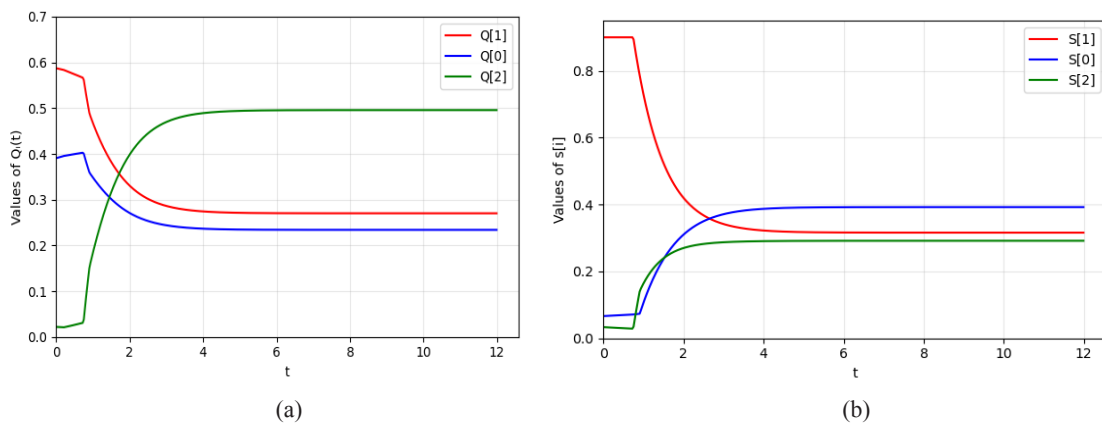
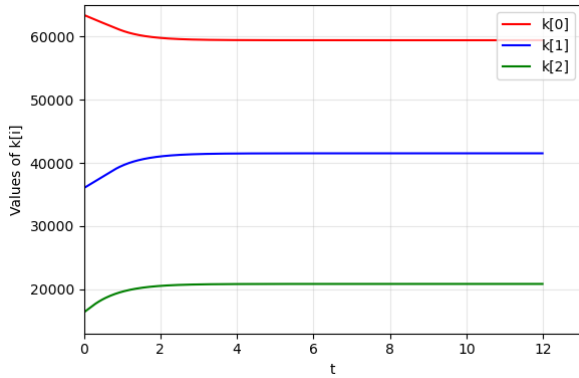
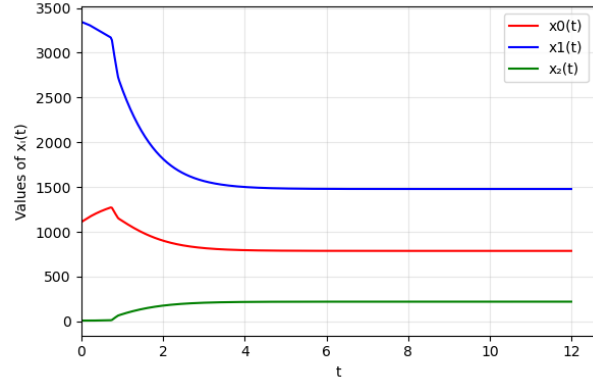


Figure 3. Optimal distribution of: (a) labor resources; (b) investment resources

In addition, during the computations, the values were obtained that determine the optimal allocation of labor resources $(\theta_0(t), \theta_1(t), \theta_2(t))$ and investment resources $(s_0(t), s_1(t), s_2(t))$ among the production, investment, and consumer subsystems, satisfying the balance relations (4)–(5). The corresponding results are presented in Figures 3 (a) and (b).



(a)



(b)

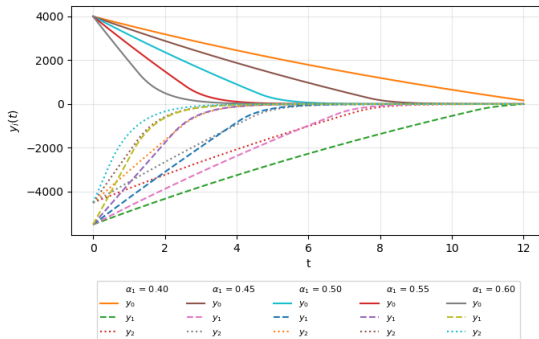
Figure 4. (a) capital intensity dynamics in the subsystems of an open economy; (b) dynamics of subsystem output in an open economy

All simulations were performed using a Runge–Kutta-type implicit integrator. The integration step was limited by a maximum step size of 0,02. Relative and absolute tolerances were set to 10^{-8} and 10^{-10} , respectively. Matrix equations were solved numerically at each time step.

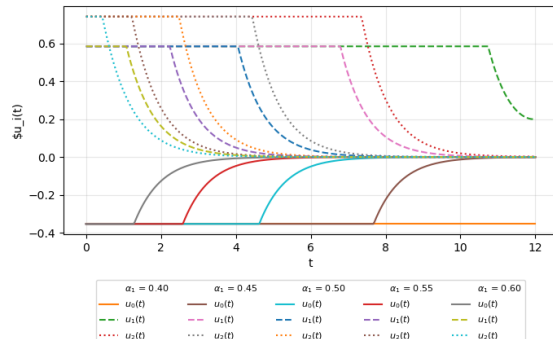
In the numerical implementation the matrix $W(t, T)$ was integrated backward in time starting from a small regularized terminal value $W(T, T) = \varepsilon I$ with $\varepsilon = 10^{-5}$, preventing singular inversion of

$W(t, T)$ near the terminal time. The full set of parameters used in the experiments is reported in Tables 1–2.

To assess the robustness of the proposed numerical procedure, additional experiments were performed under variations of model parameters and initial conditions. In particular, the elasticity coefficient of the investment subsystem α_1 was varied within the interval $[0.4, 0.6]$ with discretization step 0.05. (Figures 5 and 6).



(a)



(b)

Figure 5. System behavior under variation of the elasticity coefficient α_1 : (a) state trajectories $y_i(t)$; (b) control actions $u_i(t)$

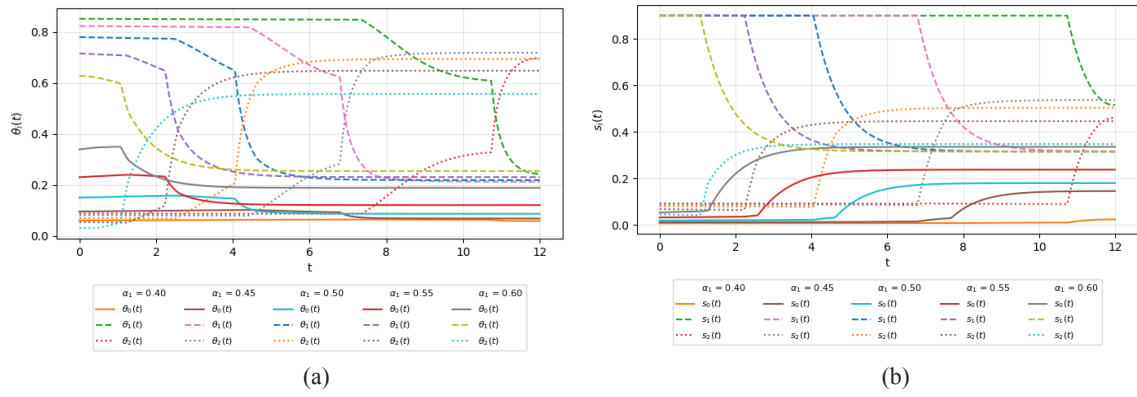


Figure 6. Resource allocation dynamics under variation of α_1 :
 (a) labor distribution $\theta_i(t)$; (b) investment shares $s_i(t)$

Furthermore, several initial deviations from the stable state were considered, including both large-scale and moderate perturbations:

$$y(t_0) = (10000, -10000, 8000)^*,$$

$$y(t_0) = (500, -500, -500)^*,$$

The obtained trajectories of states and controls are presented in Figures 7 and 8.

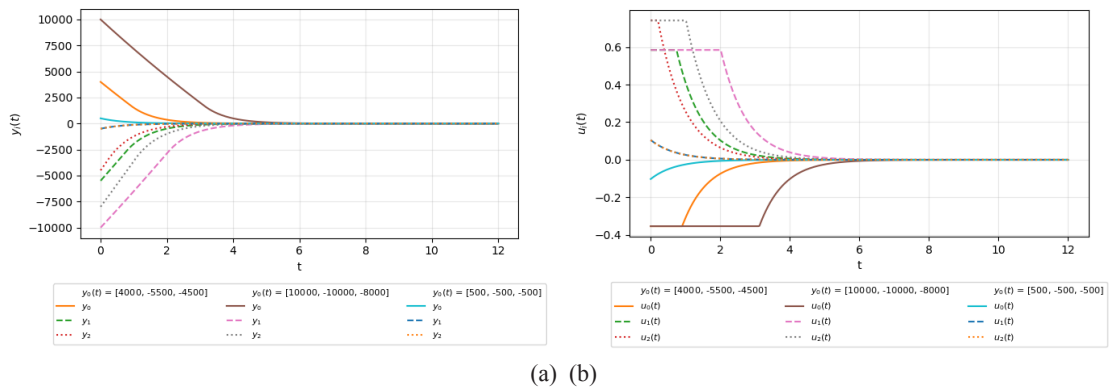


Figure 7. Robustness of the algorithm with respect to initial conditions:
 (a) state trajectories $y_i(t)$; (b) control actions $u_i(t)$

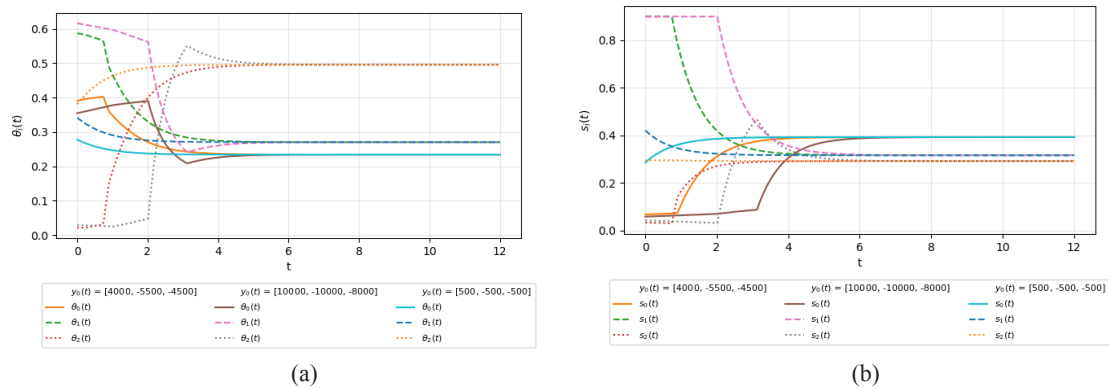


Figure 8. Resource allocation under different initial states:
 (a) labor distribution $\theta_i(t)$; (b) investment shares $s_i(t)$

For all tested configurations, the synthesized controls remained inside the admissible bounds. The system consistently approached the equilibrium neighborhood, and the qualitative structure of labor and investment allocation was preserved.

4. Discussion

The developed algorithm has demonstrated efficiency in the numerical implementation of the optimal control problem under multiple constraints. Its main advantage lies in transforming the original constrained formulation into an equivalent problem that is more convenient for computation. As a result, stable solutions were obtained that reflect the

patterns of resource allocation among the economic subsystems.

Unlike general-purpose optimization methods, which require solving high-dimensional nonlinear programming problems, the proposed algorithm constructs admissible controls through the integration of matrix differential equations combined with projection formulas, which simplifies the numerical implementation. The computational performance of the approach was evaluated by comparing it with standard nonlinear programming solvers (Table 3).

As shown, the proposed method achieves comparable values of the objective functional while requiring significantly less computational time.

Table 3. Stationary state of an open economic system

Method	Runtime (s)	Objective value $J(y, u)$
Proposed algorithm	0.11	3.51×10^7
NLP (SLSQP)	3.32	3.52×10^7
NLP (L-BFGS-B)	3.17	3.52×10^7

Numerical experiments showed that labor resources are concentrated in the consumer subsystem, as it is the most labor-intensive. Investments are primarily directed to the production and investment subsystems, due to their role in maintaining production capacities, renewing capital, and developing infrastructure. These results are consistent with economic logic and confirm the validity of the proposed algorithm.

From an economic perspective, the obtained allocation structure reflects the functional roles of the subsystems within an open economy. The concentration of labor in the consumer subsystem indicates that final demand formation relies primarily on human resources, while capital accumulation mechanisms dominate in the production and investment subsystems. The redistribution pattern generated by the algorithm demonstrates that sustainable dynamics require continuous support of sectors responsible for capacity expansion and technological renewal. A reduction of investments in these subsystems would immediately slow down the growth of capital intensity and, consequently, limit future output levels.

Therefore, the computed trajectories can be interpreted not only as a mathematical solution but also as an indicator of structurally necessary proportions between consumption and

accumulation. They provide quantitative guidance for policy design aimed at maintaining long-term stability while avoiding excessive resource diversion toward short-term consumption. The algorithm not only makes it possible to determine the stable state of the system but also to investigate the influence of model parameters on the structure of the optimal allocation. This capability makes it a valuable tool for analyzing alternative scenarios and strategies of economic development.

During the numerical simulations, additional verification of the admissibility conditions was performed at each integration step. In particular, the positivity of the state variables, boundedness of trajectories, and feasibility of the share parameters within the interval (0,1) were monitored. The balance relations were also checked numerically and were satisfied up to machine precision along the computed trajectories. No violations of admissibility were detected in the reported experiments.

5. Conclusions

This paper presents an algorithm for the numerical implementation of the optimal resource control problem in an open economy model. The algorithm is based on the Lagrange multipliers framework and is intended to obtain solutions under labor, investment, and foreign trade constraints.

The computational experiments demonstrate that the system converges to a stable regime that ensures coordinated resource allocation and sustainable development dynamics. The practical significance of the study lies in extending classical optimal control techniques to an open-economy environment with explicit consideration of external trade flows and industrial security restrictions. Thus, the proposed algorithm can be used for quantitative analysis and modeling of resource allocation dynamics in open economic systems.

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Author Contributions

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Conflicts of Interest

The authors declare no conflict of interest.

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